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Spinup Dynamics of Gyrostats

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Attention is given to the spinup dynamics of gyrostats containing a single axisymmetric rotor. Spinup of the rotor is due to a small constant torque applied by a motor on the platform. The dynamics are described by four first-order differential equations, which are put into noncanonical Hamiltonian form. Using conservation of angular momentum and the method of averaging, the equations of motion are reduced to a single scalar first-order equation for the slow evolution of the Hamiltonian. This reduction is formally valid for small spinup torques and in regions of phase space where the unperturbed motion is periodic. The unperturbed separatrices are therefore regions where averaging fails to describe the motion adequately and are also indicative of dramatic changes in the attitude dynamics. Exact solutions to the averaged equation are used to justify further the projection of solutions of the four-dimensional system onto the plane of the slow states. The reduced two-dimensional slow state space is used to construct a single planar diagram that is useful for portraying spinup dynamics.

Introduction

MOST artificial satellites contain one or more spinning rotors to provide gyroscopic stability of a desired orientation or attitude of the vehicle. Dual-spin spacecraft use the spin of a large rotor to maintain pointing accuracy of a comparatively small Earth-pointing antenna platform. Bias momentum satellites, on the other hand, use small but rapidly spinning momentum wheels to control the attitude of a large platform. In any case, such a vehicle is put into orbit in an all-spun condition, in which the rotors are locked and the vehicle spins as a single body. After the mission orbit is acquired, an attitude acquisition maneuver is performed in which a combination of external torques from thrusters and internal torques from spinup motors is used to achieve the mission orientation. Some satellites may also regularly perform attitude maneuvers using external and internal torques. In this paper we consider rotational motion during a spinup maneuver using internal torques only.

Our model consists of a rigid-body satellite containing a single rigid axisymmetric rotor constrained to relative rotation about its symmetry axis. This model is usually called a gyrostat. The gyrostat is free of external torques, and the rotor is spun up by a small constant internal torque applied by the platform. We use a noncanonical Hamiltonian formulation of the rotational equations of motion to analyze this spinup problem. We also indicate how to apply our approach to multiple-rotor gyrostats.

The equations of motion for gyrostats have two well-known integrable cases: the constant-speed rotor case and the zero-axial-torque case. In apparently the only study of a natural gyrostat, Volterra[1] introduced a gyrostat model in order to explain precession of Earth's equinoxes and obtained a solution for the angular velocities in terms of Weierstrass's elliptic function. More recently, and motivated by artificial satellites, Masaitis[2] obtained a solution in terms of Jacobi's elliptic functions for the angular velocities of an axial gyrostat (rotor parallel to principal axis) with no axial torque. Interestingly, he suggested using the solution as a basis for a perturbation solution for more complicated problems. At about the same time, Leipholz,[3] motivated by flight of aerospace vehicles containing rotating parts, also found the solution for this case. Leimanis[4] gave an extensive treatment apparently based on Masaitis[2] and Leipholz.[3] Wittenburg[5] treated more general cases where the rotor is not parallel to any principal axis of the gyrostat. For the axial gyrostat, he solved for the angular velocities in terms of Jacobi's elliptic functions. For the more general cases, he reduced the equations of motion to a quadrature but did not evaluate the integral. Cochran et al.[6] extended the previous results for axial gyrostats, obtaining solutions for the Euler angles in terms of elliptic integrals. Cochran and Shu[7] obtained similar results for an axisymmetric gyrostat with two constant-speed rotors.

Spinup problems for spacecraft with rotors have been investigated by numerous authors. Kane[8] studied axisymmetric gyrostats, as did Sen and Bainum,[9] who also used a perturbation analysis to handle slight misalignment of the rotor. Anchev[10] developed the required spinup torques to reorient a three-rotor gyrostat from the rigid-body gravity gradient equilibrium to one of the known gyrostat equilibria. Gebman and Mingori[11] used matched asymptotic expansions to obtain an approximate solution for flat spin recovery of axial gyrostats. Barba and Aubrun[12] introduced the use of momentum spheres to study momentum transfer attitude dynamics of axial gyrostats; they also noted the significance of the time rate of change of energy during momentum transfer. Kane[13] and Chen and Kane[14] investigated the spinup dynamics of a zero-momentum gyrostat. Hubert[15,16] studied spinup of a gyrostat with two rotors: one a constant-speed rotor, the other subject only to a viscous damping torque. He gave an effective description of the dual-spin turn using the momentum sphere approach,[15] and he showed that the dual-spin condition is asymptotically stable for all initial conditions, provided that the constant-speed rotor spins fast enough.[16] Guelman[17] developed an attitude recovery control law for axial gyrostats. Kinsey et al.[18] used the method of averaging to study nutation growth due to rotor imbalance. Hall and Rand[19] used the method of averaging applied to the elliptic function solution for axial gyrostats to study spinup dynamics and developed a new approach based on the slow variation of the Hamiltonian during spinup. Their approach was used to explain the nutation growth phenomenon known as precession phase lock[20] and extended to include gyrostats with two rotors.[21] Tsui and Hall[22] used this approach to study unbalanced gyrostats of the type studied by Kinsey et al.,[18] and Mazzoleni et al.[23] applied the method to dual spinners with flexible appendages. In this paper, we extend the results of Ref. 19 to include more general gyrostats and to make use of a noncanonical Hamiltonian formulation.

Noncanonical Hamiltonian formulations have been applied to several problems involving rotating systems. This approach is especially useful for finding equilibria and characterizing their stability. A basic reference for this material is the text by Olver.[24] The first application of this approach to gyrostat dynamics is apparently due to Krishnaprasad,[25] wherein he used a similar formulation to determine stability criteria for gyrostats with driven and damped free-spinning rotors.

The noncanonical approach has been applied extensively by Maddocks and his colleagues,[26–28] primarily to determine stability of equilibria. In Maddocks[26] the heavy top is used as an example to illustrate the stability analysis of a Hamiltonian system.

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In Wang et al.,[27] the rigid-body satellite is studied in detail, and in Maddocks and Sachs[28] a detailed account is given of stability results for both canonical and noncanonical Hamiltonian systems.

We begin by developing the equations of motion, arriving at a Hamiltonian formulation for the dimensionless angular momenta. This formulation lends itself to easy determination of the first integrals and equilibria of the motion and sets the stage for applying the method of averaging.[29] A useful bifurcation diagram is obtained by graphing the equilibrium values of the Hamiltonian H as a function of the rotor angular momentum μ, and spinup dynamics trajectories are presented as projections onto the μH plane. Averaging is used to justify projection of solutions onto the μH plane, which is the slow state space for the system. Exact solutions to the averaged equation governing the slow states further illustrate the usefulness of this approach. Finally, we briefly discuss the case of a gyrostat with multiple rotors.

Equations of Motion

Here we adopt much of Hughes' notation.[30] In the first approximation, a dual-spin spacecraft is usually modeled as two rigid bodies: an asymmetric unbalanced body $\mathcal{P}$ and an axisymmetric balanced body $\mathcal{R}$, connected by a shaft that allows relative rotation about the symmetry axis of $\mathcal{R}$, which is fixed in $\mathcal{P}$. We denote the system as $\mathcal{P}+\mathcal{R}$ (see Fig. 1). An internal motor is used to provide an equal and opposite torque to each body along the connecting shaft. Usually $\mathcal{P}$ is the platform and $\mathcal{R}$ is the rotor; however, some authors study spinup where the platform is axisymmetric and the rotor is either unbalanced or asymmetric.[18,20,22]

Referring to Fig. 1, we define the following symbols:

$\mathcal{F}_i$ = inertial reference frame

$\mathcal{F}_b$ = reference frame fixed in $\mathcal{P}$

$\hat{\boldsymbol{b}}_i$ = orthonormal basis for $\mathcal{F}_b$, $i = 1, 2, 3$

$\boldsymbol{O}$ = mass center of $\mathcal{P}+\mathcal{R}$ and origin of $\mathcal{F}_b$

$\vec{a}$ = axis of relative rotation, fixed in $\mathcal{F}_b$

$\vec{\tilde{\omega}}$ = angular velocity of $\mathcal{P}$ relative to $\mathcal{F}_i$

$\tilde{\omega}_s$ = angular velocity of $\mathcal{R}$ relative to $\mathcal{F}_b$

$\vec{\vec{\tilde{I}}}$ = moment of inertia tensor of $\mathcal{P}+\mathcal{R}$ about $\boldsymbol{O}$

$\tilde{I}_s$ = moment of inertia of $\mathcal{R}$ about $\vec{a}$

The tilde is used to denote dimensional variables and parameters. Below we will nondimensionalize the equations.

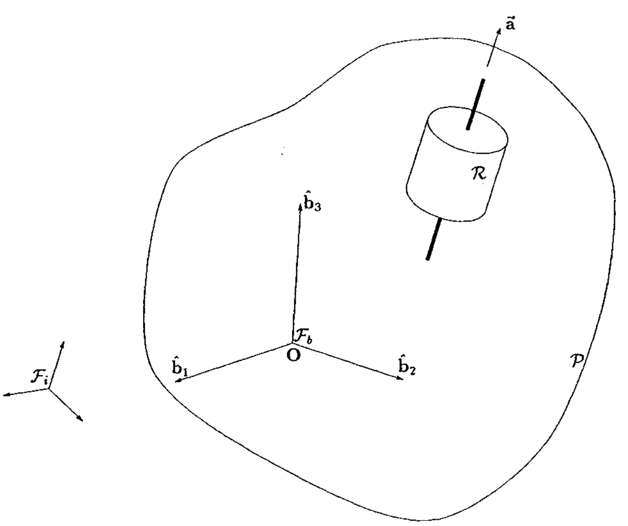

Fig. 1 Gyrostat model of dual-spin spacecraft.

Some other notational conventions are established here. In general, vectors and tensors are expressed in $\mathcal{F}_b$, so that $\boldsymbol{a}$ is the column matrix of $\vec{a}$ and $\tilde{\boldsymbol{I}}$ is the 3×3 matrix of $\vec{\vec{\tilde{I}}}$. Note that we do not assume $\mathcal{F}_b$ is a principal frame, so $\tilde{\boldsymbol{I}}$ is not necessarily diagonal. In fact, a more convenient choice for $\mathcal{F}_b$ is the frame in which $\tilde{\boldsymbol{I}} - \tilde{I}_s\boldsymbol{a}\boldsymbol{a}^T$ is diagonal. The notation $\boldsymbol{a}^\times$ denotes the skew-symmetric matrix form of a vector, i.e.,

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Leftrightarrow \boldsymbol{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{1}$$

so that $\boldsymbol{a}^\times\boldsymbol{b}$ is the matrix equivalent of $\vec{a}\times\vec{b}$.

The torque-free rotational motion equations for the angular momentum of $\mathcal{P}+\mathcal{R}$ are

$$\frac{\mathrm{d}\tilde{\boldsymbol{h}}}{\mathrm{d}\tilde{t}} = -\tilde{\boldsymbol{\omega}}^\times\tilde{\boldsymbol{h}} \tag{2}$$

$$\frac{\mathrm{d}\tilde{h}_a}{\mathrm{d}\tilde{t}} = \tilde{g}_a \tag{3}$$

where

$\tilde{\boldsymbol{h}}$ = angular momentum of $\mathcal{P}+\mathcal{R}$ expressed in $\mathcal{F}_b$, $= \tilde{\boldsymbol{I}}\tilde{\boldsymbol{\omega}} + \tilde{h}_s\boldsymbol{a}$

$\tilde{h}_a$ = angular momentum of $\mathcal{R}$ about $\vec{a}$, $= \tilde{I}_s\boldsymbol{a}^T\tilde{\boldsymbol{\omega}} + \tilde{h}_s$

$\tilde{h}_s$ = angular momentum of $\mathcal{R}$ relative to $\mathcal{F}_b$, $= \tilde{I}_s\tilde{\omega}_s$

$\tilde{g}_a$ = torque applied to $\mathcal{R}$ by $\mathcal{P}$ about $\vec{a}$

$\tilde{t}$ = time

Using the definitions of $\tilde{\boldsymbol{h}}$ and $\tilde{h}_a$, we eliminate $\tilde{\boldsymbol{\omega}}$ from Eq. (2), giving

$$\frac{\mathrm{d}\tilde{\boldsymbol{h}}}{\mathrm{d}\tilde{t}} = -\left[\tilde{\boldsymbol{J}}^{-1}(\tilde{\boldsymbol{h}} - \tilde{h}_a\boldsymbol{a})\right]^\times\tilde{\boldsymbol{h}} \tag{4}$$

or, equivalently,

$$\frac{\mathrm{d}\tilde{\boldsymbol{h}}}{\mathrm{d}\tilde{t}} = \tilde{\boldsymbol{h}}^\times\left[\tilde{\boldsymbol{J}}^{-1}(\tilde{\boldsymbol{h}} - \tilde{h}_a\boldsymbol{a})\right] \tag{5}$$

where

$$\tilde{\boldsymbol{J}} \triangleq \tilde{\boldsymbol{I}} - \tilde{I}_s\boldsymbol{a}\boldsymbol{a}^T \tag{6}$$

Note that the transformation $\tilde{\boldsymbol{I}} \leftrightarrow \tilde{\boldsymbol{J}}$, $\tilde{h}_s \leftrightarrow \tilde{h}_a$ is the transformation that relates zero-spinup-torque gyrostats ($\tilde{h}_a$ = const) to constant-rotor-speed gyrostats ($\tilde{h}_s$ = const).[31] Since $\tilde{\boldsymbol{J}}^{-1}$ appears in both Eqs. (4) and (5), it is apparent that, by choosing a frame in which $\tilde{\boldsymbol{J}}$ is diagonal, the equations of motion will be simplified. Although a nonprincipal frame is usually undesirable, in this paper we develop appropriate theory to describe spinup in terms of scalar variables that are frame invariant, namely the Hamiltonian and the rotor angular momentum.

In both Eqs. (4) and (5), the 3×3 matrix $[\cdot]^\times$ on the right-hand side is skew symmetric. In fact, both equations are of the form

$$\frac{\mathrm{d}\tilde{\boldsymbol{h}}}{\mathrm{d}\tilde{t}} = \mathcal{J}(\tilde{\boldsymbol{h}})\nabla\tilde{H}(\tilde{\boldsymbol{h}}) \tag{7}$$

where $\mathcal{J}(\tilde{\boldsymbol{h}})^T = -\mathcal{J}(\tilde{\boldsymbol{h}})$, $\tilde{H}(\tilde{\boldsymbol{h}})$ is a Hamiltonian function, and the gradient $\nabla\tilde{H}(\tilde{\boldsymbol{h}})$ is the column matrix of partial derivatives of $\tilde{H}(\tilde{\boldsymbol{h}})$ with respect to the components of $\tilde{\boldsymbol{h}}$. Provided that $\mathcal{J}(\tilde{\boldsymbol{h}})$ satisfies Jacobi's identity, which is easily verified for both Eqs. (4) and (5), Eq. (7) is in noncanonical Hamiltonian form.[24]

For Eq. (4), the Hamiltonian is actually one-half the square of the angular momentum vector magnitude; i.e.,

$$\tilde{H} = \tfrac{1}{2}\tilde{\boldsymbol{h}}^T\tilde{\boldsymbol{h}} \tag{8}$$

whereas for Eq. (5), the Hamiltonian is

$$\tilde{H} = \tfrac{1}{2}\tilde{\boldsymbol{h}}^T\tilde{\boldsymbol{J}}^{-1}\tilde{\boldsymbol{h}} - \tilde{h}_a\boldsymbol{a}^T\tilde{\boldsymbol{J}}^{-1}\tilde{\boldsymbol{h}} \tag{9}$$

which differs from the rotational kinetic energy by a constant. Note that in the latter form the Hamiltonian also depends on time, since $\tilde{h}_a$ depends on time due to the spinup torque $\tilde{g}_a$. The skew-symmetric matrices $\mathcal{J}(\tilde{h})$ are also different for the two formulations, and in Eq. (4), $\mathcal{J}(\tilde{h})$ also depends on $\tilde{h}_a$. Since there are two similar Hamiltonian formulations, this system is actually bi-Hamiltonian,[24] but we do not pursue this here.

We use the latter form [Eq. (5)] because the Hamiltonian is time varying, which we exploit in the reduction by averaging below. In this case $\mathcal{J}(\tilde{h}) = \tilde{h}^{\times}$, and unless $\tilde{h} \equiv 0$, its null space $\mathcal{N}(\tilde{h}^{\times})$ is spanned by $\tilde{h}$. We assume throughout that $\tilde{h} \neq 0$. This has the immediate consequence that the magnitude of the angular momentum vector ($\tilde{h}^2 = \tilde{h}^T\tilde{h}$) is constant. Although this fact is well known, it is worthwhile to see how it results from Eq. (5). Define

$$\tilde{C} = \tfrac{1}{2}\tilde{h}^T\tilde{h} \tag{10}$$

The time derivative of $\tilde{C}$ is

$$\frac{d\tilde{C}}{d\tilde{t}} = \nabla\tilde{C}^T\frac{d\tilde{h}}{d\tilde{t}} = \tilde{h}^T\mathcal{J}(\tilde{h})\nabla\tilde{H} = -\nabla\tilde{H}^T\mathcal{J}(\tilde{h})\tilde{h} = 0 \tag{11}$$

Thus $\tilde{C}$ is constant because $\nabla\tilde{C} = \tilde{h}$ is in the null space of $\mathcal{J}(\tilde{h}) = \tilde{h}^{\times}$. This type of first integral is independent of the Hamiltonian and is called a Casimir function.[24]

Since $\nabla\tilde{C}$ is in the null space of $\mathcal{J}(\tilde{h})$, the addition of a function of $\tilde{C}$ to the Hamiltonian has no effect on the equations of motion. Thus any Hamiltonian that puts Eq. (5) into the form of Eq. (7) may be expressed as

$$\tilde{H} = \tilde{H}(\tilde{h}, \tilde{t}) = \tfrac{1}{2}\tilde{h}^T\tilde{J}^{-1}\tilde{h} - \tilde{h}_a(\tilde{t})a^T\tilde{J}^{-1}\tilde{h} + f(\tilde{C}) \tag{12}$$

where $f(\tilde{C})$ is an arbitrary function of the first integral in Eq. (10). Note that the energylike constant y in Ref. 19 is essentially $\tilde{H}$ with $f(\tilde{C}) = \tilde{C}/(\tilde{I}_1 - \tilde{I}_s)$. The Hamiltonian is time varying due to the spinup torque in Eq. (3) and satisfies

$$\frac{d\tilde{H}}{d\tilde{t}} = \frac{\partial\tilde{H}}{\partial\tilde{h}_a}\frac{d\tilde{h}_a}{d\tilde{t}} = -\tilde{g}_a a^T\tilde{J}^{-1}\tilde{h} \tag{13}$$

Thus $\tilde{H} = \text{const}$ if $\tilde{g}_a = 0$, and $\tilde{H}$ is slowly varying if $\tilde{g}_a$ is "small."

Before proceeding, we nondimensionalize by introducing dimensionless angular momentum x, rotor momentum μ, Hamiltonian H, and time t, defined by

$$\begin{aligned} \tilde{h} &= \tilde{h}x & \tilde{H} &= \tilde{h}^2 H/\tilde{I}_c \\ \tilde{h}_a &= \tilde{h}\mu & \tilde{t} &= \tilde{I}_c t/\tilde{h} \end{aligned} \tag{14}$$

where $\tilde{I}_c$ is a characteristic moment of inertia, such as $\operatorname{tr}\tilde{I}$ or $\tilde{I}_s$. All moments of inertia are nondimensionalized by dividing by $\tilde{I}_c$. In addition, the dimensionless torque ε is defined by

$$\tilde{g}_a = \tilde{h}^2\varepsilon/\tilde{I}_c \tag{15}$$

Under this transformation, the equations of motion (2) and (3) become

$$\dot{x} = x^{\times}\nabla H \tag{16}$$

$$\dot{\mu} = \varepsilon \tag{17}$$

where the ∇ operator is now with respect to the dimensionless angular momentum x, and the Hamiltonian H is

$$H = H(x, t) = \tfrac{1}{2}x^T J^{-1}x - \mu(t)\alpha^T x + f(C) \tag{18}$$

which satisfies

$$\dot{H} = \varepsilon\frac{\partial H}{\partial\mu} = -\varepsilon\alpha^T x \tag{19}$$

Note that we have abbreviated $a^T J^{-1}$ by α^T, which is constant but is not a unit vector. Conservation of angular momentum becomes

$$C = \tfrac{1}{2}x^T x = \tfrac{1}{2} \tag{20}$$

Equation (16) is a noncanonical Hamiltonian system of equations. If $\varepsilon = 0$, it is both autonomous and integrable. If $\varepsilon = \text{const} \neq 0$, Eq. (17) is directly integrable, and Eq. (16) is nonautonomous since H depends on μ.

Equations describing the kinematics may be combined with Eq. (16) to form a sixth-order noncanonical system. Specifically, Poisson's equations[4,26] for the body frame components of a unit vector $\vec{k}$ in an inertially fixed direction may be written as

$$\dot{k} = -\omega^{\times}k \tag{21}$$

However, since we have assumed torque-free motion, we can take $\vec{k}$ in the direction of $\vec{h}$, in which case Poisson's equations are identical to Eq. (16). In this paper we consider only the system given by Eq. (16), with μ varying according to Eq. (17).

We now develop the scalar equations corresponding to Eq. (16). The choice of $f(C) = -C/J_1$, together with definitions of dimensionless inertia parameters i_2 and i_3 by

$$i_2 = (J_1 - J_2)/(J_1J_2) \qquad i_3 = (J_1 - J_3)/(J_1J_3) \tag{22}$$

leads to the Hamiltonian

$$H = \tfrac{1}{2}\left(i_2x_2^2 + i_3x_3^2\right) - \mu(\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3) \tag{23}$$

angular momentum magnitude

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{24}$$

and the three differential equations

$$\begin{aligned} \dot{x}_1 &= (i_3 - i_2)x_2x_3 + \mu(\alpha_2x_3 - \alpha_3x_2) \\ \dot{x}_2 &= -i_3x_1x_3 + \mu(\alpha_3x_1 - \alpha_1x_3) \\ \dot{x}_3 &= i_2x_1x_2 + \mu(\alpha_1x_2 - \alpha_2x_1) \end{aligned} \tag{25}$$

Note that if we had chosen a frame in which $\tilde{I}$ is diagonal, there would be several more terms in each equation. Also note that when $\mu = 0$, Eqs. (25) are equivalent to Euler's equations for a rigid body with inertia matrix J.

Example. Here and throughout, we use a specific example for illustration. Our fictitious spacecraft has principal moments of inertia $\tilde{I}_1 = 100$, $\tilde{I}_2 = 70$, $\tilde{I}_3 = 40$, and the rotor's axial moment of inertia is $\tilde{I}_s = 15$. The rotor axis direction in this principal frame is given by $a = (\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$. If we use $\operatorname{tr}\tilde{I}$ for the characteristic moment of inertia $\tilde{I}_c$, then the rotation matrix taking vectors from the principal frame to the "pseudo-principal" frame is

$$Q = \begin{bmatrix} 0.9867 & -0.1476 & -0.0687 \\ 0.1330 & 0.9741 & -0.1829 \\ 0.0939 & 0.1714 & 0.9807 \end{bmatrix} \tag{26}$$

The pseudo-principal moments of inertia are $J_1 = 0.4576$, $J_2 = 0.3107$, $J_3 = 0.1602$, the components of a are $(0.4448, 0.5335, 0.7194)^T$, and the components of α are $(0.9720, 1.7170, 4.4897)^T$. The inertia parameters are $i_2 = 1.0328$ and $i_3 = 4.0558$.

Integrable $\varepsilon = 0$ Problem

The $\varepsilon = 0$ problem is well known to be completely integrable, as described in the introduction. The text by Hughes[30] discusses the geometry of the $\varepsilon = 0$ solutions for the angular momenta as intersections of the ellipsoid and sphere defined by Eqs. (18) and (20). It is interesting to note that our choice of $f(C)$ means the Hamiltonian as expressed in Eq. (23) defines an elliptic paraboloid rather than an ellipsoid. In the context of noncanonical Hamiltonian systems, integrability may be formally shown using Theorem 6.35 of Olver.[24] The purpose of this section is to state some of these known results in the present notation. Specifically, we need the results regarding equilibrium points and periodic solutions of Eq. (16).

First we recall that, for $\varepsilon = 0$, solutions to Eq. (16) are periodic and bounded, with the exception of a finite number of separatrices connecting saddle points on the momentum sphere. The separatrices are bounded but have infinite period. There are either two, four, or six equilibrium points on the momentum sphere, depending on the value of μ. For examples of momentum spheres, see Barba and Aubrun,[12] Hubert,[15] Hughes,[30] Hall and Rand,[19] or Hall.[20,21]

The equilibrium point analysis is based on the fact that equilibrium solutions x_e of Eq. (16) must satisfy $\nabla H(x_e) \in \mathcal{N}(x_e^\times)$. Since $\mathcal{N}(x^\times)$ is spanned by $x = \nabla C(x)$, the requirement for x_e to be an equilibrium is

$$\nabla H(x_e) = \lambda \nabla C(x_e) \tag{27}$$

where λ is a Lagrange multiplier and C is the Casimir function associated with conservation of angular momentum, Eq. (20).

The calculation of equilibria is accomplished by noting that solutions to Eq. (27) are critical points of the function

$$F(x) = H(x) - \lambda C(x) \tag{28}$$

Furthermore, $H(x)$ also depends on μ, which is constant for $\varepsilon = 0$, and we need to compute equilibria for a range of values of μ. Thus we use Euler–Newton continuation[32] to vary μ and compute the critical points of $F(x)$. Whereas a rigorous stability analysis could be carried out,[28] we make use of the fact that for $\mu = 0$ the Hamiltonian reduces to that of a fictitious rigid body with inertia matrix J and easily characterized equilibria. For example, if $J_1 > J_2 > J_3$, then $x = (\pm 1, 0, 0)^T$ corresponds to stable steady rotations about the major (or oblate) axis of the fictitious rigid body. Similarly, $x = (0, \pm 1, 0)^T$ and $x = (0, 0, \pm 1)^T$ correspond to unstable and stable rotations about the intermediate and minor (or prolate) axes, respectively. We use these equilibria as the starting points in the Euler–Newton continuation algorithm, in which case the stability of the branches is evident.

The equilibrium branches depend only on μ and the constant parameters J and α. For each μ there are multiple branches, corresponding in general to different values of H, denoted H_e. Thus we can consider x_e and H_e as functions of μ. In Fig. 2, we show the components of $x_e(\mu)$ for the example gyrostat defined in the previous section. The stable branches (centers) are identified with solid curves, and dashed curves are used to denote the unstable branches (saddles). We also use the following notation to identify the six equilibrium branches: $O_\mu^\pm$, $P_\mu^\pm$, and $U_\mu^\pm$, where O, P, and U denote oblate, prolate, and unstable, respectively; the $\pm$ corresponds to which $\mu = 0$ equilibrium the branch starts at, and μ indicates that the equilibrium depends on μ. Thus, for example, $O_\mu^\pm$ denotes the equilibrium branch starting at $x = (+1, 0, 0)^T$.

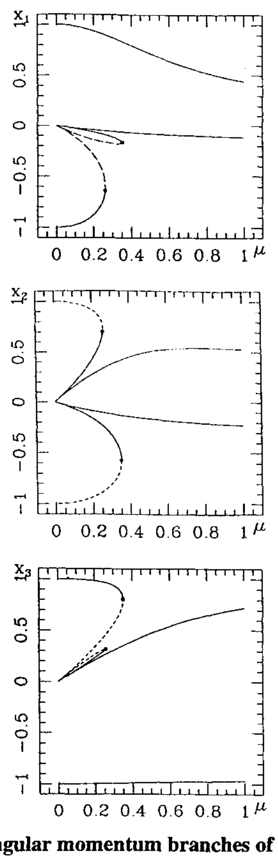

Fig. 2 Angular momentum branches of equilibria.

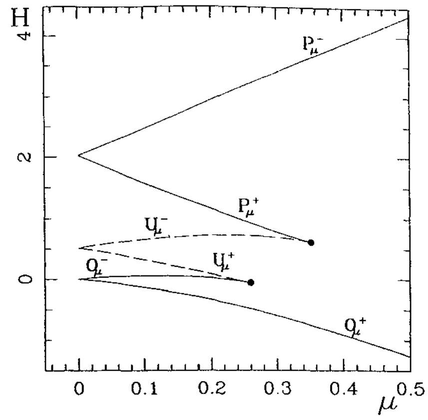

Fig. 3 μH plane.

A useful bifurcation diagram is constructed by plotting the branches of $H_e(\mu)$, as shown in Fig. 3. The solid curves correspond to centers on the momentum spheres, and the dashed curves correspond to saddles. By comparison with Fig. 2, it can be seen that the two bifurcation points, where a dashed curve and a solid curve meet and disappear, correspond to fold bifurcations, where a saddle-center pair coalesce. This type of bifurcation diagram was first introduced for axial gyrostats,[19] which only admit pitchfork bifurcations. The μH plane has the basic form shown in Fig. 3 for every gyrostat of the type studied here. For example, the μy planes used for axial gyrostats[19,20] and for "biaxial" gyrostats[21,22] are special cases of the μH plane introduced here.

Other bifurcation diagrams for depicting equilibria as a function of rotor speed have been based on the constant-speed rotor[25,30,33,34] and usually plot a rational function of λ vs h^2/h_s^2. The advantage of the μH plane is that it also represents the slow state space for spinup dynamics, so that spinup can be shown on a single plot. This is illustrated in the following section and rigorously justified in the subsequent section on averaging.

As noted, when $\varepsilon = 0$, $x(t)$ is periodic and may be written as

$$x(t) = x_0(\phi) \tag{29}$$

where $\phi \in [0, 1)$ is an angular phase variable defined by $\phi \triangleq t/T$, where T is the period of the solution, and $t \in [0, T)$ is time measured along the periodic trajectory. Since the period depends on H and μ, it is clear that ϕ satisfies

$$\dot{\phi} = \Omega_0(H, \mu) \triangleq 1/T \tag{30}$$

In principle $x_0(\phi)$ can be expressed in terms of Jacobi's elliptic functions and the frequency $\Omega_0(H, \mu)$ in terms of complete elliptic integrals. This has been carried out in detail for the axial gyrostat, which has 13 different cases.[35] In the more general cases considered here, the elliptic integrals are less tractable.[5,34] Note that the frequency $\Omega_0(H, \mu)$ is constant, since H and μ are constants for $\varepsilon = 0$. Thus a point in the μH plane corresponds to a particular periodic solution, which may be interpreted as a closed polhode-like curve on the momentum sphere. Of course this correspondence is not quite unique, since two curves on the sphere could have the same value of H. For example, it is evident from Fig. 3 that, for $\mu = 0$, both O_μ^+ and O_μ^- have the same value of H, and it follows that periodic oscillations about these equilibria may have the same values of H. Usually a particular application will remove this ambiguity. On the separatrices the solution is not periodic or, rather, has infinite period. Thus, the frequency $\Omega_0(H, \mu)$ vanishes for (H, μ) corresponding to a separatrix.

Spinup

As noted in the introduction, a satellite is initially deployed in an all-spun condition; i.e., the rotor is locked and the vehicle spins

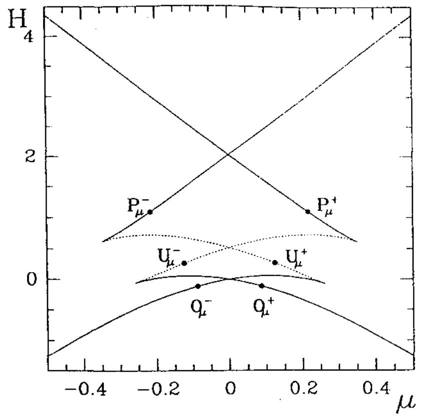

Fig. 4 μH **plane with all-spun equilibria shown.**

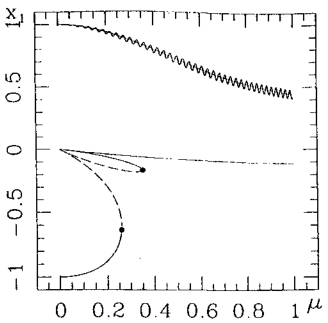

Fig. 5 Angular momentum branch x_1 **with spinup trajectory for** O_μ^C**.**

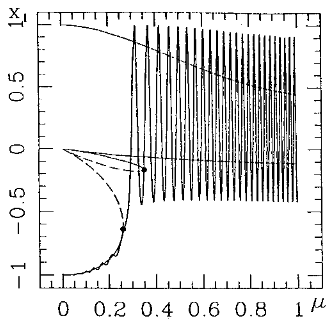

Fig. 6 Angular momentum branch x_1 **with spinup trajectory for** O_μ^-**.**

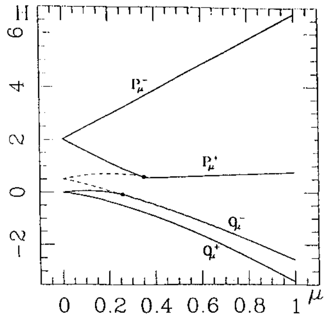

Fig. 7 μH **plane with spinup trajectories.**

as a single rigid body. The spinup motor is turned on to spin up the rotor and despin the platform. In this section we develop these ideas in terms of the equations of motion for spinup of gyrostats. We also use the μH plane to illustrate spinup trajectories. In the following section, we use the method of averaging to rigorously justify projection of spinup trajectories onto the μH plane.

Initial conditions for spinup are typically taken to be near a stable equilibrium in the all-spun condition. The all-spun condition is defined by $\tilde{\omega}_s = 0$, which leads to an expression for the all-spun rotor momentum:

$$\mu_{\text{as}} = I_s \boldsymbol{a}^T \boldsymbol{I}^{-1} \boldsymbol{x} \tag{31}$$

Note that a constraint torque on the rotor is required since $\dot{\mu}_{\text{as}} \neq 0$, unless the gyrostat is in equilibrium. Thus the all-spun condition is not in general an $\varepsilon = 0$ trajectory. However, since platform and rotor are rigid, this case corresponds to rigid-body motion, which is also a well-known integrable problem. To compute μ_{as}, we use the principal frame of the gyrostat, since $\boldsymbol{I}$ is diagonal, and the rigid-body equilibria are simple in this frame. Thus there are four stable initial conditions corresponding to pure spin about either the major or minor axis of the all-spun body, and two unstable initial conditions corresponding to pure spin about the intermediate axis of the all-spun body. For our example, the all-spun equilibria correspond to $\pm$ the columns of $\boldsymbol{Q}$ given in Eq. (26) and are shown in Fig. 4. Note that the $\boldsymbol{O}_\mu^-$, $\boldsymbol{U}_\mu^-$, and $\boldsymbol{P}_\mu^-$ all-spun initial conditions correspond to $\mu < 0$ and that the μH plane is symmetric about $\mu = 0$.

A definition for the conclusion of spinup depends on the desired attitude motion of the gyrostat. For example, if we require an inertially fixed platform ($\omega = \boldsymbol{0}$), then the final value of the dimensionless rotor momentum is $\mu_f = 1$. Since this is approximately the case for many applications, we use $\mu_f = 1$ as the definition for conclusion of spinup. Once $\mu = 1$ is attained, the motor is turned off ($\varepsilon = 0$), and the ensuing motion behaves as described in the previous section.

Continuing with our example, we show two spinup trajectories in Figs. 5–7. In Figs. 5 and 6, we show the x_1 component of the angular momentum during spinup. To aid in understanding, we overlay the equilibrium branches of Fig. 2. The trajectory shown in Fig. 5 starts near the $\mu = 0$ equilibrium point for $\boldsymbol{O}_\mu^+$, denoted $\boldsymbol{O}_0^+$, and the trajectory shown in Fig. 6 starts near the $\mu = 0$ equilibrium point for $\boldsymbol{O}_\mu^-$, denoted $\boldsymbol{O}_0^-$. As expected, the trajectory that starts near the disappearing equilibrium branch leads to large-amplitude oscillations.

In Fig. 7, we show H vs μ for the four spinup trajectories starting near $\boldsymbol{O}_\mu^\pm$ and $\boldsymbol{P}_\mu^\pm$, with the bifurcation diagram of Fig. 3 overlaid. From Figs. 5 and 6 it is clear that showing all four spinup trajectories in the angular momentum plots would be quite cluttered and not very useful, whereas in the μH plane the four trajectories do not intersect. This is because initial conditions in the μH plane almost (but not quite) lead to unique trajectories in the plane. The H vs μ trajectories closely follow the H_e vs μ curves they start near. We show in the next section that the $H_e(\mu)$ curves are exact solutions to the averaged equation, so that this result is expected.

Note that H is calculated using Eq. (18) whereas Eqs. (16) and (17) are numerically integrated. Exactly the same results may be obtained by simultaneously integrating Eq. (19).

Averaging

In this section we generalize earlier results,[19,21,36] obtaining a single first-order differential equation describing the flow in the μH plane. We also show that the equilibrium branches $H_e(\mu)$ are exact solutions to the averaged equation. Furthermore, the results are stated in terms of the Hamiltonian and therefore apply to any system whose equations of motion may be expressed in the form of Eqs. (16) and (17), regardless of the Hamiltonian.

We begin by putting the equations of motion into the correct form for averaging.[29] This is based on applying variation of parameters to the $\varepsilon = 0$ solution discussed earlier. Thus for $\varepsilon = 0$ the solution $\boldsymbol{x}(t)$ may be expressed as $\boldsymbol{x}_0(\phi; H, \mu)$, where H and μ are constant "parameters." We assume the $\varepsilon \neq 0$ solution has the same form; i.e., $\boldsymbol{x}(t) = \boldsymbol{x}_0(\phi, H, \mu)$, where H and μ are now variables satisfying Eqs. (19) and (17). We still need an equation for ϕ, which will take the form of Eq. (30) when $\varepsilon = 0$. Straightforward differentiation leads to the desired equation, and the equations of motion (16) and (17) may be replaced by

$$\dot{\phi} = \Omega_0(H, \mu) + \varepsilon \Omega_1(\phi, H, \mu) \tag{32}$$

$$\dot{H} = -\varepsilon \boldsymbol{\alpha}^T \boldsymbol{x}_0(\phi, H, \mu) \tag{33}$$

$$\dot{\mu} = \varepsilon \tag{34}$$

where $\Omega_0(H, \mu)$ is the $\mathcal{O}(1)$ frequency and $\varepsilon \Omega_1(\phi, H, \mu)$ is the "small" phase-dependent drift caused by spinup. These equations clearly separate the fast and slow dynamics; that is, ϕ is the rapidly varying phase and H and μ are slowly varying. Recall that $\Omega_0(H, \mu)$

vanishes when (H, μ) corresponds to a separatrix of the unperturbed system; therefore ϕ is not rapidly varying in a neighborhood of an instantaneous separatrix.

Since $x(\phi, H, \mu)$ is periodic when $\varepsilon = 0$, the right-hand sides of Eqs. (33) and (34) are also periodic, with period 1 in ϕ, and we average them over one period. Equation (34) is unaffected by averaging, and the averaged version of Eq. (33) is

$$\dot{\bar{H}} = -\varepsilon\alpha^T\bar{x}(H, \mu) \tag{35}$$

where

$$\bar{x}(H, \mu) \triangleq \int_0^1 x_0(\phi; H, \mu)\,\mathrm{d}\phi \tag{36}$$

and $\bar{H}$ denotes the approximation to H. Equation (35) may be combined with $\dot{\mu} = \varepsilon$ to obtain

$$\bar{H}' = -\alpha^T\bar{x}(H, \mu) \tag{37}$$

where $(\cdot)' \triangleq \mathrm{d}(\,)/\mathrm{d}\mu$, thus giving a single approximate equation for the slow evolution of the Hamiltonian during spinup. This reduction of Eqs. (16) and (17) allows us to study the four-dimensional spinup problem as a trajectory in the μH plane. The averaging theorem states that solutions to Eq. (37) remain within $\mathcal{O}(\varepsilon)$ of solutions to Eqs. (16) and (17) for time scales of $\mathcal{O}(1/\varepsilon)$. Thus the fast and slow dynamics are partially decoupled by averaging.

In principle, one could evaluate the integral in Eq. (36) and numerically or approximately integrate Eq. (37) independent of additional equations of motion. This has been done for the axial gyrostat,[19] and approximate closed-form solutions were obtained.[35] A similar approach led to approximate closed-form solutions for axisymmetric gyrostats with flexible appendages.[23] For the general gyrostats considered here, the elliptic integrals are not as tractable and we do not obtain approximate solutions. Rather, we simply use the formal validity of Eq. (37) to justify projection of numerical solutions of the exact equations of motion onto the μH plane. Thus the μH plane is a simple graphical tool for depicting spinup dynamics.

The usefulness of the μH plane is further illustrated by the fact that the solid curves in the plane (i.e., the branches of stable equilibria) are integral curves of Eq. (37). The proof is that Eq. (37) may be written as $\bar{H}' = \overline{\partial H/\partial \mu}$, and at an equilibrium $\overline{\partial H/\partial \mu} = \overline{\partial H_e/\partial \mu} = \partial H_e/\partial \mu$. And since $H_e(\mu)$ depends only on μ, $\partial H_e/\partial \mu = \mathrm{d}H_e/\mathrm{d}\mu$, so the stable branches of $H_e(\mu)$ solve Eq. (37). The unstable branches do not, since for those branches the period is infinite and the required averages are undefined. Since the branches of stable equilibria are solutions to the averaged equation for spinup, and since averaged solutions remain within $\mathcal{O}(\varepsilon)$ of exact solutions, exact spinup trajectories starting near a stable branch will remain near that branch. This is confirmed by the trajectories in Fig. 7 and explains our earlier observation. These results do not hold for larger values of ε or for trajectories near the unperturbed separatrices.

As noted above, the unperturbed solution has infinite period at the separatrices; hence averaging is not valid in a neighborhood of the separatrices. Of course there are no separatrices in the perturbed system, but it is evident that the perturbed trajectories undergo dramatic changes near the "instantaneous" separatrix crossings. In the μH plane spinup trajectories follow the stable branches until an unstable branch is reached, at which point an instantaneous separatrix crossing occurs. Then the trajectory runs approximately in a straight line. It is at this point that large-amplitude oscillations begin, and the amplitude remains approximately constant after the instantaneous separatrix crossing occurs (cf. Figs. 5 and 6).

Multiple-Rotor Gyrostats

In the case of a gyrostat with N rotors, the equations of motion may be written as

$$\dot{x} = x^{\times}\nabla H \tag{38}$$

$$\dot{\mu}_j = \varepsilon_j \qquad j = 1, N \tag{39}$$

where μ_j is the axial angular momentum of the jth rotor, and ε_j is the axial torque applied by the platform on this rotor. The Hamiltonian is

$$H = \frac{1}{2}x^T J^{-1} x - \sum_{j=1}^{N} \mu_j \alpha_j^T x + f(C) \tag{40}$$

which satisfies

$$\dot{H} = \sum_{j=1}^{N} \varepsilon_j \frac{\partial H}{\partial \mu_j} \tag{41}$$

Here $\alpha_j = J^{-1}a_j$ and $J = I - \sum_{j=1}^{N} I_{sj} a_j a_j^T$.

Assuming the ε_j are all constant, averaging leads to

$$\dot{\bar{H}} = -\sum_{j=1}^{N} \varepsilon_j \alpha_j^T \bar{x}(H, \mu) \tag{42}$$

along with Eqs. (39). Thus the slow state space is the $(N + 1)$-dimensional space spanned by the μ_j and H. For a particular spinup problem where all the ε_j are fixed constants, averaged spinup trajectories will be in one planar slice of this space.[21] Another case of interest would be for some of the rotors to be driven by spinup torques, whereas others are viscously damped, and the rest are driven by constant-speed motors. We do not consider these cases in this paper.

Conclusions

The attitude dynamics of rigid gyrostats are described by four ordinary differential equations that are put into a third-order noncanonical Hamiltonian form. When there is a spinup torque acting on the rotor, these equations are nonautonomous and are not integrable. When the spinup torque is small and constant, the equations of motion may be rigorously reduced to a single approximate first-order nonautonomous equation describing the slow evolution of the Hamiltonian as a function of rotor momentum. This reduction is based on conservation of angular momentum and application of the method of averaging. The geometric interpretation of the averaged equation leads to a simple plane diagram that is useful both as a bifurcation diagram for the unperturbed problem and as an approximate state space for spinup trajectories.

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